**Ph.D.Thesis of Salahaddin University-Erbil Academic Staff Studied Abroad**

**Title of thesis:** Analytic and Algebraic Aspects of Integrability for First Order Partial Differential Equations

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**Abstract:**

This work is devoted to investigating the algebraic and analytic integrability of first order polynomial partial differential equations via an understanding of the well-developed area of local and global integrability of polynomial vector fields.

In the view of characteristics method, the search of first integrals of the first order partial differential equations

$$P(x,y,z)\frac{\partial z(x,y)}{\partial x}+Q(x,y,z)\frac{\partial z(x,y)}{\partial y}=R(x,y,z), \tag{1}$$

is equivalent to the search of first integrals of the system of the ordinary differential equations

$$\frac{dx}{dt}=P(x,y,z), \qquad \frac{dy}{dt}=Q(x,y,z), \qquad \frac{dz}{dt}=R(x,y,z). \tag{2}$$

The trajectories of (2) will be found by representing these trajectories as the intersection of level surfaces of first integrals of (1).

We would like to investigate the integrability of the partial differential equation (1) around a singularity. This is a case where understanding of ordinary differential equations will help understanding of partial differential equations. Clearly, first integrals of the partial differential equation (1) are first integrals of the ordinary differential equations (2). So, if (2) has two first integrals $\emptyset_1(x,y,z)=C_1$ and $\emptyset_2(x,y,z)=C_2$, where $C_1$ and $C_2$ are constants, then the general solution of (1) is $F(\emptyset_1,\emptyset_2)=0$, where $F$ is an arbitrary function of $\emptyset_1$ and $\emptyset_2$.

We choose for our investigation a system with quadratic nonlinearities and such that the axes planes are invariant for the characteristics: this gives three dimensional Lotka–Volterra systems

$$\dot{x}=\frac{dx}{dt}=P=x(\lambda+ax+by+cz),$$
$$\dot{y}=\frac{dy}{dt}=Q=y(\mu+dx+ey+fz),$$
$$\dot{z}=\frac{dz}{dt}=R=z(\nu+gx+hy+kz),$$

where $\lambda,\mu,\nu\neq 0$.

Several problems have been investigated in this work such as the study of local integrability and linearizability of three dimensional Lotka–Volterra equations with $(\lambda:\mu:\nu)$–resonance. More precisely, we give a complete set of necessary and sufficient conditions for both integrability and linearizability for three dimensional Lotka-Volterra systems for (1 : -1 : 1), (2 : -1 : 1) and (1 : -2 : 1)–resonance. To prove their sufficiency, we mainly use the method of Darboux with the existence of inverse Jacobi multipliers, and the linearizability of a node in two variables with power-series arguments in the third variable. Also, more general three dimensional systems have been investigated and necessary and sufficient conditions are obtained. In another approach, we

also consider the applicability of an entirely different method which based on the monodromy method to prove the sufficiency of integrability of these systems.

These investigations, in fact, mean that we generalized the classical centre-focus problem in two dimensional vector fields to three dimensional vector fields. In three dimensions, the possible mechanisms underling integrability are more difficult and computationally much harder.
We also give a generalization of Singer's theorem about the existence of Liouvillian first integrals in codimension 1 foliations in $\mathbb{C}^n$ as well as to three dimensional vector fields.

Finally, we characterize the centres of the quasi-homogeneous planar polynomial differential systems of degree three. We show that at most one limit cycle can bifurcate from the periodic orbits of a centre of a cubic homogeneous polynomial system using the averaging theory of first order.
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